Question de cours

- Opérateur hermitique $A = A^{\dagger}$ et $\langle \varphi | A | \chi \rangle = \langle \chi | A | \varphi \rangle^*$
- Opérateur de projection $P_i = |u_i\rangle\langle u_i|$ et $\sum_{i=1}^{N} P_i = 1$
- $E_n^{(1)} = \langle \varphi_n^0 | W | \varphi_n^0 \rangle$

Opérateur d'évolution d'un spin 1/2.

$$\vec{\Pi} = \gamma \vec{S} \qquad \vec{B}_0 \begin{vmatrix} -\frac{\omega_x}{\gamma} \\ -\frac{\omega_y}{\gamma} \\ -\frac{\omega_3}{\gamma} \end{vmatrix}$$

1) $U(t) = e^{-\frac{iWt}{\hbar}}$ avec $W = -\vec{\Pi}\cdot\vec{B}_0 = -\gamma\, \vec{S}\cdot\vec{B}_0$

$$= -\gamma\,(S_x B_x + S_y B_y + S_3 B_3)$$
$$= (\omega_x S_x + \omega_y S_y + \omega_3 S_3)$$

soit $\Pi = \frac{1}{\hbar}(\omega_x S_x + \omega_y S_y + \omega_3 S_3)$

$$\Pi = \frac{1}{2}\left[\omega_x \sigma_x + \omega_y \sigma_y + \omega_3 \sigma_3\right] = \frac{1}{2}\left[\begin{pmatrix} 0 & \omega_x \\ \omega_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\omega_y \\ i\omega_y & 0 \end{pmatrix} + \begin{pmatrix} \omega_3 & 0 \\ 0 & -\omega_3 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} \omega_3 & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_3 \end{pmatrix}$$

$$\Pi^2 = \frac{1}{4}\begin{pmatrix} \omega_3 & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_3 \end{pmatrix}\begin{pmatrix} \omega_3 & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \omega_3^2 + \omega_x^2 + \omega_y^2 & 0 \\ 0 & \omega_x^2 + \omega_y^2 + \omega_3^2 \end{pmatrix}$$

$$= \frac{1}{4}(\omega_x^2 + \omega_y^2 + \omega_3^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{\omega_0}{2}\right)^2$$

2) $U(t) = e^{-i\Pi t} = \cos(\Pi t) - i \sin(\Pi t)$

$$\cos(\Pi t) = \sum_m \frac{(-1)^m (\Pi t)^{2m}}{(2m)!} = \sum_m (-1)^m \left(\frac{\omega_0}{2}\right)^{2m} \frac{t^{2m}}{(2m)!} = \cos\left(\frac{\omega_0 t}{2}\right)$$

$$\Pi \sin(\Pi t) = \Pi \sum_m (-1)^m \frac{(\Pi t)^{2m+1}}{(2m+1)!} = \sum_m (-1)^m \frac{(\Pi t)^{2m+2}}{t\,(2m+1)!} = \sum_m \frac{(-1)^m (\Pi t)^{2(m+1)}}{t\,(2m+1)!}$$

$$= \sum_m \frac{(-1)^m t^{2(m+1)}}{t\,(2m+1)!}\left(\frac{\omega_0}{2}\right)^{2(m+1)} = \frac{\omega_0}{2}\sum_m \frac{(-1)^m \left(\frac{\omega_0 t}{2}\right)^{2m+1}}{(2m+1)!} = \frac{\omega_0}{2}\sin\left(\frac{\omega_0 t}{2}\right)$$

donc $U(t) = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i\omega_0}{2\Pi}\sin\left(\frac{\omega_0 t}{2}\right) = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i\omega_0 \Pi}{2\underbrace{\Pi^2}_{=\omega_0^2/4}}\sin\left(\frac{\omega_0 t}{2}\right)$

$$= \cos\left(\frac{\omega_0 t}{2}\right) - \frac{2i\Pi}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right) \quad \text{cqfd.}$$

Matrice $U(t)$

$$U(t) = \cos\left(\frac{\omega_0 t}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\begin{pmatrix} \omega_3 & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_3 \end{pmatrix}$$

$$U(t) = \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 & -\frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)(\omega_x - i\omega_y) \\ -\frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)(\omega_x + i\omega_y) & \cos\left(\frac{\omega_0 t}{2}\right) + \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 \end{pmatrix}$$

3) $|\psi(0)\rangle = |+\rangle$; $|\psi(t)\rangle = U(t)|+\rangle = \alpha|+\rangle + \beta|-\rangle$

$$\mathcal{P}_{++}(t) = |\langle + | \psi(t)\rangle|^2 = |\langle + | U(t) | + \rangle|^2 \text{ cqfd.}$$

calcul de $U(t)|+\rangle = \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 \\ -\frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)(\omega_x + i\omega_y) \end{pmatrix}$

$$\langle + | U(t) | + \rangle = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 .$$

$$\Rightarrow \quad \mathcal{P}_{++}(t) = |\langle + | U(t) | + \rangle|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right) + \sin^2\left(\frac{\omega_0 t}{2}\right)\left(\frac{\omega_3}{\omega_0}\right)^2$$

$$= \left(1 - \sin^2\left(\frac{\omega_0 t}{2}\right)\right) + \left(\frac{\omega_3}{\omega_0}\right)^2 \sin^2\left(\frac{\omega_0 t}{2}\right)$$

$$= 1 - \frac{(\omega_x^2 + \omega_y^2)}{\omega_0^2}\sin^2\left(\frac{\omega_0 t}{2}\right)$$

$$P_{max} = 1 - \frac{(\omega_x^2 + \omega_y^2)}{\omega_0^2}$$