L3 SPC – corrigé Thermostat mai 2012

I 1) $$\underbrace{\varepsilon = \frac{p^2}{2m}}_{\text{energie cinétique}} - \underbrace{\frac{m\omega^2\rho^2}{2}}_{\text{energie potentielle}}$$

energie potentielle

energie cinétique

2) $$z = \frac{1}{h^3}\iiint d^3\vec{r}\, e^{\frac{\beta m\omega^2\rho^2}{2}} \iiint d^3\vec{p}\, e^{-\frac{\beta p^2}{2m}}$$

$$= \frac{1}{h^3}\iiint_{\mathcal{D}} d^3\vec{r}\, e^{\frac{\beta m\omega^2\rho^2}{2}} \left(\int_{-\infty}^{\infty} dp\, e^{-\frac{\beta p^2}{2m}}\right)^3$$

3) $$z = \frac{1}{h^3}(2\pi m k_B T)^{3/2} \times \iiint \rho\, d\rho\, d\varphi\, dz\, e^{+\frac{\beta m\omega^2\rho^2}{2}}$$

$$z = \underbrace{\frac{1}{h^3}(2\pi m k_B T)^{3/2} \times V}_{z_c} \times \underbrace{\frac{1}{V}\int_0^{2\pi} d\varphi \int_0^L dz \int_0^R \rho\, e^{\frac{\beta m\omega^2\rho^2}{2}} d\rho}_{z_p}$$

$$z = z_c \times \frac{1}{V}\, 2\pi L \int_0^R e^{\frac{\beta m\omega^2\rho^2}{2}} \rho\, d\rho$$

$$\boxed{V = \pi R^2 L}$$

$$u = \frac{\rho^2}{2} \Rightarrow \frac{2}{2}\rho\, d\rho = du$$

$$z = z_c \times \frac{2\pi L}{V}\int_0^{R^2/2} e^{\beta m\omega^2 u}\, du$$

$$= z_c \times \frac{2\pi L}{V}\, \frac{1}{\beta m\omega^2}\left(e^{\frac{\beta m\omega^2 R^2}{2}} - 1\right)$$

$$z = z_c \times \frac{\pi L R^2}{V}\, \frac{2}{\beta m\omega^2 R^2}\left(e^{\frac{\beta m\omega^2 R^2}{2}} - 1\right)$$

$$\boxed{z = \left(\frac{V}{\Lambda^3}\right) \left(\frac{e^{\beta K} - 1}{\beta K}\right)}$$

4) $$Z = \frac{z^N}{N!} = \underbrace{\frac{1}{N!}\left(\frac{V}{\Lambda^3}\right)^N}_{Z_c} \underbrace{\left(\frac{e^{\beta K} - 1}{\beta K}\right)^N}_{Z_p} = Z_c Z_p$$

5) $\langle E\rangle = -\dfrac{\partial \ln Z}{\partial \beta} = -\dfrac{\partial \ln Z_c}{\partial \beta} - \dfrac{\partial \ln Z_p}{\partial \beta}$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\left[\ln\left(\frac{V}{\Lambda^3}\right)^N - N\ln N + N\right] - \frac{\partial}{\partial\beta}\left[\ln\frac{(e^{\beta K}-1)^N}{\beta K}\right]$$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\left[\ln\left(V^N \times \left(\frac{2\pi m}{\beta h^2}\right)^{3N/2}\right)\right] - \frac{\partial}{\partial\beta}\left[\ln(e^{\beta K}-1)^N\right] + \frac{\partial}{\partial\beta}\left[\ln(K\beta)^N\right]$$

$$= -\frac{3N}{2}\times\frac{\partial}{\partial\beta}\left[\ln\left(\frac{1}{\beta}\right)\right] - N\left(\frac{Ke^{\beta K}}{e^{\beta K}-1}\right) + N\frac{\partial}{\partial\beta}\left[\ln\beta\right]$$

$$= \frac{3N}{2}\frac{1}{\beta} - N\left(\frac{Ke^{\beta K}}{e^{\beta K}-1}\right) + N\frac{1}{\beta}$$

$$\langle E\rangle = \frac{3}{2}Nk_BT + \left\{Nk_BT - N\left(\frac{Ke^{\beta K}}{e^{\beta K}-1}\right)\right\}$$

$$\langle E\rangle = \underbrace{\frac{3}{2}Nk_BT}_{E_c} + \underbrace{Nk_BT\left\{1 - \frac{Ke^{\beta K}}{e^{\beta K}-1}\right\}}_{E_p}$$

$\omega \to 0 \Rightarrow K \to 0$

$$\langle E\rangle = \frac{3}{2}Nk_BT + Nk_BT\underbrace{\left\{1 - \frac{\beta K}{\beta K}\right\}}_{\to 0}$$

$$\lim_{\omega\to 0}\langle E\rangle = \frac{3}{2}Nk_BT$$

$(e^{\beta K} - 1 \simeq \beta K)$

(gaz parfait non soumis à une force centrifuge)

6) $n(\rho) = A\, e^{\beta K \frac{\rho^2}{R^2}}$

$N = $ nb total d'atomes $= \iiint_V n(\rho)\, dV$

$$N = A\iiint_V n(\rho)\,\rho\, d\rho\, d\varphi\, dz = 2\pi L A\int_0^R n(\rho)\,\rho\, d\rho$$

$$N = 2\pi L A\int_0^R e^{\beta K\frac{\rho^2}{R^2}}\rho\, d\rho \qquad \begin{cases} u = \rho^2 \\ du = 2\rho\, d\rho \end{cases}$$

$$N = \frac{A\,2\pi L}{2}\int_0^{R^2} e^{\beta K \frac{u}{R^2}}\,du = \pi L A R^2 \frac{(e^{\beta K}-1)}{\beta K}$$

$$A = \frac{N}{\pi R^2 L}\,\frac{\beta K}{e^{\beta K}-1} = \frac{N}{V}\,\frac{\beta K}{e^{\beta K}-1}$$

$$\boxed{n(\rho) = \frac{N}{V}\,\frac{\beta K}{e^{\beta K}-1}\,e^{\beta K\frac{\rho^2}{R^2}}}$$

$\omega \to 0$, $K \to 0$

$$n(\rho) \simeq \frac{N}{V}\,\beta K\,\frac{e^{\beta K\frac{\rho^2}{R^2}}}{\beta K} \simeq \frac{N}{V}\,\underbrace{e^{\beta K\frac{\rho^2}{R^2}}}_{\text{tend vers } 1}$$

$$\lim_{\omega\to 0} n(\rho) = \frac{N}{V} = cte$$

N molécules de gaz parfait d~~ont~~ répartie de façon homogène dans $V = \pi R^2 L$

7) $p(\rho)V = N(\rho)k_BT$

$$p(\rho) = n(\rho)k_BT = \frac{N}{V}\,K\,\frac{e^{\beta K\rho^2/R^2}}{(e^{\beta K}-1)}$$

$$p(\rho) = \frac{N}{\pi \cancel{R^2} L}\,\frac{m\omega^2\cancel{R^2}}{2}\,\frac{e^{\beta K\rho^2/R^2}}{(e^{\beta K}-1)}$$

$$\boxed{p(\rho) = \frac{N}{2\pi L}\,m\omega^2\,\frac{e^{\beta K\rho^2/R^2}}{(e^{\beta K}-1)}}$$

$\rho \to 0$ $\quad p(0) = \dfrac{N m\omega^2}{2\pi L}\,\dfrac{1}{(e^{\beta K}-1)}$

$$\boxed{p(\rho) = p(0)\,e^{\beta K\rho^2/R^2}}$$

$\omega \to 0$; $\left(e^{\beta K\frac{\rho^2}{R^2}} \simeq \beta K\frac{\rho^2}{R^2} + 1\right) \to 1$

$$\lim_{\omega\to 0} p(\rho) = p(0) = \frac{N m\omega^2}{2\pi L}\,\frac{1}{\beta K} \simeq \frac{\cancel{N m\omega^2}}{\cancel{2}\pi L}\,\frac{\cancel{2}k_BT}{\cancel{m\omega^2}R^2}$$

$$\lim_{\omega\to 0} p(\rho) = \frac{N k_BT}{\pi R^2 L} = \frac{N k_BT}{V} \to$$ gaz parfait en l'absence de force centrifuge

II

1) $\vec{J}_n(x,t) = -D \dfrac{dn(x,t)}{dx} \vec{e}_x$

coefficient de diffusion ($m^2 s^{-1}$)

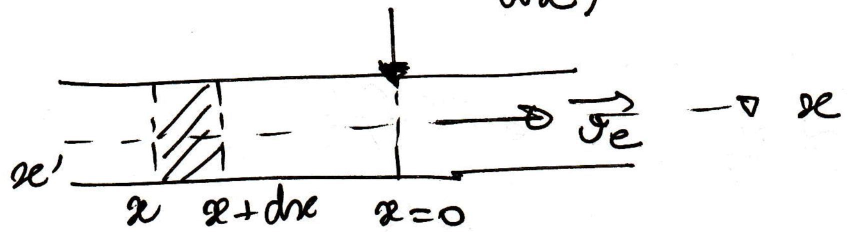
vecteur courant volumique ($m^{-2} s^{-1}$)

gradient de densité particulaire (m^{-4})

$\vec{J}_{n_e} = n \vec{v}_e = n v_e \vec{e}_x$

$\vec{J}_{n_{total}} = \left(n v_e - D \dfrac{\partial n}{dx} \right) \vec{e}_x$

2)



$dN = \left[J_{n_t}(x,t) - J_{n_t}(x+dx,t) \right] S\, dt$

$= -\dfrac{\partial J_{n_t}}{\partial x} S\, dt\, dx$

$\dfrac{dN}{dx\, S\, dt} = -\dfrac{\partial J_{n_t}}{\partial x} = \dfrac{\partial n}{\partial t}$

$$\boxed{\frac{\partial n}{\partial t} = +D \frac{\partial^2 n}{\partial x^2} - \frac{\partial n}{\partial x} v_e}$$

3) $D \dfrac{d^2 n}{dx^2} - \dfrac{dn}{dx} v_e = 0$ $\quad \boxed{\dfrac{D}{v_e} \dfrac{d^2 n}{dx^2} - \dfrac{dn}{dx} = 0}$

$\dfrac{D}{v_e} \dfrac{dn}{dx} - n = A$

$(x < 0)$

$\dfrac{dn}{dx} - n \dfrac{v_e}{D} = 0$

$n = K e^{\frac{v_e}{D} x}$

$n(x) = K e^{\frac{v_e}{D} x} + A \dfrac{v_e}{D}$

$x \to -\infty$, $n(-\infty) = K e^{\frac{v_e}{D} x} + \dfrac{A v}{D}$

$= 0$

$(x<0)$ $\quad n(-\infty) = 0 = \dfrac{A v}{D} \Rightarrow A = 0$

$x = 0 \quad n(0) = n_0 = K$

$$\boxed{n(x) = n_0 e^{\frac{v_e}{D} x}}$$

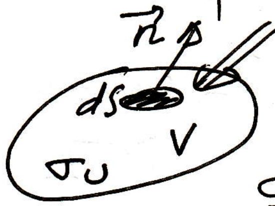
$x < 0$

III 1) $\vec{J}_u(\vec{r},t) = -\lambda\, \overrightarrow{\text{grad}}\, T$ → gradient de température ($K\,m^{-1}$)

courant volumique d'énergie interne ($W\,m^{-2}$)

conductivité thermique ($W\,m^{-1}K^{-1}$)

2)



$$\rho u = \frac{U}{V}$$

$$\begin{cases} \vec{J}_u = -\lambda \dfrac{\partial T}{\partial r}\vec{e}_r \\ I_U = \displaystyle\iint_S \vec{J}_U \cdot \vec{n}\, dS \end{cases}$$

$$\frac{dU}{dt} = \frac{d}{dt}\iiint_V (\rho u)\, dV$$

$$\frac{\partial U^r}{\partial t} = \iint_S (-\vec{J}_u)\cdot\vec{n}\, dS \; ; \quad \frac{\partial U^p}{\partial t} = \iiint_V \sigma_u\, dV$$

$$\frac{dU}{dt} = \frac{d}{dt}\iiint_V \rho u\, dV = \iiint_V -\operatorname{div}\vec{J}_u\, dV + \iiint_V \sigma_u\, dV$$

$$\boxed{\frac{\partial \rho u}{\partial t} = -\operatorname{div}\vec{J}_u + \sigma_u}$$

en régime stationnaire $-\operatorname{div}\vec{J}_u + \sigma_u = 0$

3)

$$\begin{cases} I_U = \sigma_u V = \sigma_u \cancel{2\pi} \pi R^2 L \\ \vec{J}_u = \displaystyle\iint_S \vec{J}_U\cdot\vec{n}\, dS = 2\pi r L J_U \end{cases} \qquad \boxed{\vec{n} = \vec{e}_r}$$

$$\begin{cases} -\operatorname{div}(\lambda\, \overrightarrow{\text{grad}}\, T) + \sigma_U = 0 \\ \lambda \Delta T + \sigma_U = 0 \end{cases}$$

$$T(r) \Rightarrow \lambda\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right\} = -\sigma_U$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{-\sigma_U}{\lambda} \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{-\sigma_U}{\lambda} r$$

$$r\frac{\partial T}{\partial r} = \frac{-\sigma_u}{2\lambda}r^2 + C_1 \Rightarrow \frac{\partial T}{\partial r} = \frac{-\sigma_U}{2\lambda}r + \frac{C_1}{r}$$

$\dfrac{\partial T}{\partial r}$ fini pour $r = 0 \Rightarrow C_1 = 0$

$$T(r) = -\frac{\sigma_U}{4\lambda}r^2 + C_2$$

$$T(r=R) = T_S = -\frac{\sigma_U}{4\Lambda} R^2 + C_2$$

$$C_2 = T_S + \frac{\sigma_U}{4\Lambda} R^2$$

$$T(r) = \left(T_S + \frac{\sigma_U}{4\Lambda} R^2\right) - \frac{\sigma_U}{4\Lambda} r^2 = A - Br^2$$

$$A = T_S + \frac{\sigma_U}{4\Lambda} R^2 = 443 + 4\,10^6 (2\,10^{-2})^2 = 2043\,K \;!! \;> T_f$$

$$B = \frac{\sigma_U}{4\lambda} = 4\,10^6\ K\,m^{-2}$$

l'uranium fond à 1405K ⟹ géométrie pas adaptée !

$$T(r=0) = T_{max} = A$$

$$\left(\left(\frac{\partial T(r)}{\partial r}\right)_{T=T_{max}} = 0 \Rightarrow -2Br = 0 \Leftrightarrow r = 0\right)$$

4) si $A = T_f = T_{max} \Rightarrow (T_f - T_S)\dfrac{4\Lambda}{R^2} = \sigma_{U max} = 289\ MW\,m^{-3}$

5) $\vec{J_U} = -\Lambda \dfrac{dT}{dr}\vec{e_r}$, $I_U = \sigma_U V = \sigma_U \pi (r^2 - R_i^2)$
$= J_U\, 2\pi r L$

$$J_U = \frac{\sigma_U}{2}\left(\frac{r^2 - R_i^2}{r}\right) = \frac{\sigma_U}{2}\left(r - \frac{R_i^2}{r}\right)$$

$$-\Lambda \frac{dT}{dr} = \frac{\sigma_U}{2}\left(r - \frac{R_i^2}{r}\right) \Rightarrow \frac{dT}{dr} = -\frac{\sigma_U}{2\Lambda}\left(r - \frac{R_i^2}{r}\right)$$

$$\boxed{T(r) = -\frac{\sigma_U r^2}{4\Lambda} + \frac{\sigma_U}{2\Lambda} R_i^2 \ln r + cte}$$

$$T_S = T(R_e) = T(R) = -\frac{\sigma_U R_e^2}{4\Lambda} + \frac{\sigma_U}{2\Lambda} R_i^2 \ln R_e + cte$$

$$cte = T_S + \frac{\sigma_U}{4\Lambda} R_e^2 - \frac{\sigma_U}{2\Lambda} R_i^2 \ln R_e$$

$$T(r) = \underbrace{-\frac{\sigma_U r^2}{4\Lambda}}_{B} + \underbrace{\frac{\sigma_U R_i^2}{2\Lambda}}_{C} \ln r + \underbrace{\frac{\sigma_U}{4\Lambda} R_e^2 - \frac{\sigma_U}{2\Lambda} R_i^2 \ln R_e + T_S}_{A'}$$

$$B = \frac{\sigma_U}{4\Lambda} \; ; \; C = \frac{\sigma_U}{2\Lambda} R_i^2 = 800\,K$$

$$A' = \frac{\sigma U}{4\lambda} Re^2 - \frac{\sigma U}{2\lambda} Ri^2 \ln Re + T_S$$

$$A' = 1600 + 3130 + 443 = 5173 K$$

$$T_{max} = T(r = Ri)$$

$$\left(\frac{\partial T}{\partial r} = 0 = \frac{C}{r} - 2Br \Rightarrow r^2 = \frac{C}{2B} = Ri^2 \Rightarrow r = Ri \right)$$

$$T_{max} = -\frac{\sigma U}{4\lambda} Ri^2 + \frac{\sigma U}{2\lambda} Ri^2 \ln Ri^2 + \frac{\sigma U}{4\lambda} Re^2 - \frac{\sigma U}{2\lambda} Ri^2 \ln Re + T_S$$

$$= T_S + \frac{\sigma U}{4\lambda} Re^2 - \frac{\sigma U}{4\lambda} Ri^2 - \sigma U\, Ri^2 \ln\left(\frac{Re}{Ri}\right)$$

$$= 443 + 1600 - 400 - 400 \times \ln 2$$

$$T_{max} = 1366 K < T_f$$

(l'uranium ne fond pas !)