Correction Examen Janvier 2011
Thermodynamique statistique

I A)

1) $P_J = \sigma_U \times V$ = puissance dissipée par effet Joule

$$\sigma_U = \frac{P_J}{V} = \frac{RI^2}{V} = \frac{\rho_c L}{S_1} \times \frac{I^2}{S_1 L} = \frac{\rho_c I^2}{\pi^2 b_1^4}$$

$\sigma_U = 15\,502 \ \mathrm{W\,m^{-3}}$

2) $0 \leq r \leq b_1$ Loi de Fourier $\vec{J_U} = -\lambda_c \dfrac{dT}{dr}\vec{e_r}$

(J_U indépendant de ϕ et z)

3) $$\frac{\partial(\rho u)}{\partial t} = -\mathrm{div}\,\vec{J_U} + \sigma_u = 0 \quad \text{(régime stationnaire)}$$

$$-\mathrm{div}(-\lambda_c \,\mathrm{grad}\,T) + \sigma_u = 0 \quad ; \quad \lambda_c \Delta T + \sigma_U = 0$$

$$\Delta T = \left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right\} = -\frac{\sigma_U}{\lambda_c}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{r\sigma_U}{\lambda_c} \Rightarrow r\frac{\partial T}{\partial r} = \frac{-\sigma_U}{\lambda_c}\frac{r^2}{2} + A$$

$$\frac{dT}{dr} = -\frac{\sigma_U}{2\lambda_c} r + \frac{A}{r} \quad \text{avec} \begin{cases} A = 0 \\ \text{sinon } \dfrac{dT}{dr} \to \infty \text{ en } r = 0 \end{cases}$$

$$T(r) = -\frac{\sigma_U}{4\lambda_c} r^2 + T(r=0) = \frac{-\rho_c I^2}{4\pi^2 b_1^4 \lambda_c} r^2 + T_0$$

$0 \leq r \leq b_1$

$$T(r=b_1) = T_c = \underbrace{\frac{-\rho_c I^2}{4\pi^2 b_1^2 \lambda_c}}_{1\,10^{-3} \text{ négligeable}} + T_0 = T_0$$

$\dfrac{dT}{dr}$ ds le conducteur négligeable car conducteur thermique parfait ($390 \ \mathrm{W\,m^{-1}\,K^{-1}}$)

4) $I'_U(r) = \iint_{S_2} \vec{J}'_U \cdot \vec{n}\, dS = J'_U\, 2\pi r L$

avec $b_1 \le r \le b_2$ $\qquad J'_U(r) = \dfrac{I'_U(r)}{2\pi r L} = -\lambda_i \dfrac{dT}{dr}$

$$\underset{(0 \le r \le b_1)}{I_U(r)} = \underset{(b_1 \le r \le b_2)}{I'_U(r)} = P_J = \frac{\rho_c I^2}{\pi b_1^2} L$$

$$\frac{dT}{dr} = -\frac{\rho_c I^2}{\pi b_1^2} \frac{L}{\lambda_i 2\pi r L} = \frac{-\rho_c I^2}{2\pi^2 b_1^2 \lambda_i} \frac{1}{r}$$

$$T(r) = \frac{-\rho_c I^2}{2\pi^2 b_1^2 \lambda_i} \ln r + B$$

$$T(r = b_2) = T_a = -\frac{\rho_c I^2}{2\pi^2 b_1^2 \lambda_i} \ln b_2 + B$$

$$B = T_a + \frac{\rho_c I^2}{2\pi^2 b_1^2 \lambda_i} \ln b_2$$

$$\boxed{T(r) = T_a + \frac{\rho_c I^2}{2\pi^2 b_1^2 \lambda_i} \ln\left(\frac{b_2}{r}\right)}$$

5) $T(r = b_1) = T_a + \dfrac{\rho_c I^2}{2\pi^2 b_1^2 \lambda_i} \ln\left(\dfrac{b_2}{b_1}\right) = T_c$

$$\lambda_i = \frac{\rho_c I^2}{2\pi^2 b_1^2 (T_c - T_a)} \ln\left(\frac{b_2}{b_1}\right) = 0{,}1 \text{ W m}^{-1}\text{K}^{-1}$$

B

6) $P_J = \sigma_U \times V = \sigma_U \times \pi b_1^2 \times L$

↓ transmise par rayonnement de la surface latérale du fil à la surface latérale du conducteur extérieur

Bilan : $\left[\underbrace{(2\pi b_1 L)\sigma T_c^4 \times \varepsilon_1}_{\text{rayonné par le cuivre}} - \underbrace{(2\pi b_2 L)\sigma T_a^4 \varepsilon_2}_{\text{rayonné par le conducteur extérieur}}\right]$

$$\cancel{2\pi L}\sigma\{b_1\varepsilon_1 T_c^4 - b_2\varepsilon_2 T_a^4\} = \underbrace{\sigma_u(\pi b_1^2 \cancel{L})}_{\text{effet Joule } P_J}$$

Bilan indépendant de L

$$2\sigma b_1 \varepsilon_1 T_c^4 = \sigma_u b_1^2 + 2\sigma b_2 \varepsilon_2 T_a^4$$

$$T_c = \left[\frac{\sigma_u b_1}{2\sigma\varepsilon_1} + T_a^4 \frac{b_2}{b_1}\frac{\varepsilon_2}{\varepsilon_1}\right]^{1/4}$$

$T_c = 227\ K$

7) si $\varepsilon_2 = 0{,}93$ $\quad T_c = 358\ K$

II 1) $E = \underbrace{-mgx + \frac{Kx^2}{2}}_{\text{energies potentielles}}$ distribution canonique

$$dP(x) = p(x)\,dx = \frac{1}{Z} e^{-\beta(\cancel{+}mgx + \frac{Kx^2}{2})} dx$$

$$dP(x) = \frac{1}{Z} e^{-\beta(-mgx + \frac{Kx^2}{2})} dx$$

$$Z = \int_{-\infty}^{\infty} e^{-\beta\left(\frac{Kx^2}{2} - mgx\right)} dx$$

$$\frac{Kx^2}{2} - mgx = \frac{K}{2}\left(x^2 - 2\frac{mg}{K}x\right) = \frac{K}{2}\left(x - \frac{mg}{K}\right)^2 - \frac{m^2g^2}{2K}$$

$$Z = \int_{-\infty}^{\infty} e^{-\frac{\beta K}{2}\left(x - \frac{mg}{K}\right)^2} dx \quad e^{+\frac{\beta m^2 g^2}{2K}}$$

$$Z = e^{+\frac{\beta m^2 g^2}{2K}} \int_{-\infty}^{\infty} e^{-\frac{\beta K}{2} x'^2} dx' = e^{\frac{\beta m^2 g^2}{2K}} \sqrt{\frac{2\pi k_B T}{K}}$$

(4)

2) $$dP(x) = \frac{e^{-\frac{\beta m^2 g^2}{2K}}}{\sqrt{\frac{2\pi k_B T}{K}}} e^{-\beta \frac{K}{2}\left[\left(x - \frac{mg}{K}\right)^2 - \frac{m^2 g^2}{2K}\right]} dx$$

$$dP(x) = \sqrt{\frac{K}{2\pi k_B T}}\, e^{-\beta \frac{K}{2}\left(x - \frac{mg}{K}\right)^2} dx$$

$$w(x) = \frac{1}{\sqrt{2\pi \frac{k_B T}{K}}} e^{-\beta \frac{K}{2}(x - mg/K)^2}$$

$$\int_{-\infty}^{\infty} w(x)\,dx = \sqrt{\frac{K}{2\pi k_B T}} \int_{-\infty}^{\infty} e^{-a x'^2} dx' \quad \text{avec } x' = x - \frac{mg}{K}$$
$$\text{et } a = \beta\frac{K}{2}$$

$$\int_{-\infty}^{\infty} w(x)\,dx = \sqrt{\frac{\beta K}{2\pi}} \sqrt{\frac{2\pi}{\beta K}} = 1$$

$$w(x) = \frac{1}{\sqrt{2\pi \frac{k_B T}{K}}} \exp\left[-\frac{(x - mg/K)^2}{2\,\frac{k_B T}{K}}\right]$$

$$w(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(x - \langle x\rangle)^2}{2\sigma^2}\right]$$

avec $\sigma = \sqrt{\frac{k_B T}{K}}$ et $\langle x \rangle = \frac{mg}{K}$

3) $$\sigma \leq \langle x \rangle \longleftrightarrow \sqrt{\frac{k_B T}{K}} \leq \frac{mg}{K}$$

donc $m \geq \frac{\sqrt{K k_B T}}{g}$ $\quad m_{min} = \frac{\sqrt{K k_B T}}{g}$

III

1) espace des phases $\equiv \{ q^{3N}, p^{3N} ; V \equiv [0,L] \times S , L \}$

2) $E_L = E_{piston} + E_{gaz} = fL + \sum_{i=1}^{N} \frac{p_i^2(L)}{2m}$

3) $$Z_{gp}(L) = \frac{1}{h^{3N} N!} \left[\iiint e^{-\frac{\beta p^2}{2m}} d^3\vec{p}\, d^3\vec{r} \right]^N$$

$$= \frac{1}{h^{3N} N!} V^N \left(\int_{-\infty}^{\infty} e^{-\frac{\beta p_x^2}{2m}} dp_x \right)^{3N}$$

$$Z_{gp}(L) = \frac{1}{h^{3N} N!} S^N L^N \left(\frac{2\pi m}{\beta} \right)^{3N/2}$$

4) $$Z = \underbrace{\sum_{L_j} e^{-\beta f L_j}}_{\text{somme sur tous les } L_j} \underbrace{\left(\sum_{i=1,\, L_j = L}^{N} e^{-\frac{\beta p_i^2(L)}{2m}} \right)}_{\text{somme sur tous les états de gaz parfait pour } L_j = L}$$

$$Z = \int_0^{\infty} e^{-\beta f L} dL\, Z_{gp}(L)$$ (L variable continue)

$$Z = \frac{S^N}{h^{3N} N!} \left(\frac{2\pi m}{\beta} \right)^{3N/2} \int_0^{\infty} e^{-\beta f L} L^N dL$$

$u = \beta f L$, $du = \beta f\, dL$

$$Z = \frac{S^N}{h^{3N} N!} \left(\frac{2\pi m}{\beta} \right)^{3N/2} \frac{1}{(\beta f)^{N+1}} \underbrace{\int_0^{\infty} e^{-u} u^N du}_{N!}$$

$$Z = \frac{S^N}{h^{3N}} \left(2\pi m k_B T\right)^{3N/2} \left(\frac{k_B T}{f}\right)^{N+1}$$

$$= \frac{Z_{gp}(L)}{L^N} N! \left(\frac{k_B T}{f}\right)^{N+1}$$

$$\langle L \rangle = \frac{1}{Z} \sum_{L_j} L_j \, e^{-\beta[f L_j + E_{gp}(L)]}$$

$$\langle L \rangle = \frac{1}{Z} \sum_{L_j} L_j \, e^{-\beta[f L_j + \sum_{i, L_j = L} \frac{p_i^2}{2m}(L)]}$$

$$= \frac{1}{Z\beta} \sum_{L_j} - \frac{\partial}{\partial f} \left\{ e^{-\beta[f L_j + \sum_{i, L_j = L} \frac{p_i^2}{2m}(L)]} \right\}$$

$$= - \frac{1}{Z\beta} \frac{\partial}{\partial f} \left\{ \sum_{L_j} e^{-\beta[f L_j + \sum_{i, L_j = L} \frac{p_i^2}{2m}(L)]} \right\}$$

$$= - \frac{k_B T}{Z} \frac{\partial Z}{\partial f} = - k_B T \frac{\partial \ln Z}{\partial f}$$

$$\langle L \rangle = \frac{k_B T}{f} (N+1)$$