Correction Examen Jan 2008 _ Thermostat

I) gaz parfait dans un champ de pesanteur

1)

$$z_0 = \frac{1}{h^3} \iiint\!\!\!\iiint_{-\infty}^{\infty} e^{-\beta E(\vec{r},\vec{p})}\, d^3\vec{r}\, d^3\vec{p}$$

$$z_0 = \frac{1}{h^3} \iiint_{-\infty}^{\infty} dx\, dy\, dz \iiint_{-\infty}^{\infty} e^{-\beta E(\vec{p})}\, dp_x\, dp_y\, dp_z$$

$$z_0 = \frac{V}{h^3} \left(\int_{-\infty}^{\infty} e^{-\frac{\beta p_x^2}{2m}}\, dp_x \right)^3 = V\left(\sqrt{\frac{\pi 2m}{\beta}}\right)^3$$

$$z_0 = \frac{V}{h^3} (2m\pi k_B T)^{3/2}$$

$$Z_0 = \frac{z_0^N}{N!} = \frac{V^N}{h^{3N} N!} (2m\pi k_B T)^{3N/2} \propto \left(\frac{1}{\beta}\right)^{3N/2}$$

2)

$$\langle E_0 \rangle = -\frac{\partial (\ln Z_0)}{\partial \beta} = \frac{3N}{2} \frac{1}{\beta} = \frac{3N}{2} k_B T$$

$$F_0 = -k_B T \ln Z_0 = -k_B T \left\{ \ln V^N + \ln \left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2} - \ln N! \right.$$

$$F_0 = -N k_B T \left[\ln V + \frac{3}{2} \ln \left(\frac{2\pi m k_B T}{h^2}\right) - \ln N + 1 \right]$$

$$F_0 = -N k_B T \left\{ \ln\left(\frac{V}{N}\right) + \frac{3}{2} \ln\left(\frac{2\pi m k_B T}{h^2}\right) + 1 \right\}$$

$$S_0 = -\frac{\partial F_0}{\partial T}$$ ou $$F_0 = \langle E_0 \rangle - T S_0$$

$$S_0 = \frac{\langle E_0 \rangle - F_0}{T}$$

$$S_0 = \frac{3}{2} N k_B \frac{T}{T} + N k_B \frac{T}{T} \left\{ \ln\left(\frac{V}{N}\right) + \frac{3}{2} \ln\left(\frac{2\pi m k_B T}{h^2}\right) + 1 \right\}$$

$$S_0 = N k_B \left\{ \ln\left(\frac{V}{N}\right) + \frac{3}{2} \ln\left(\frac{2\pi m k_B T}{h^2}\right) + \frac{5}{2} \right\}$$

3) $C_{V_0} = \left(\dfrac{\partial \langle E_0 \rangle}{\partial T}\right)_V = \dfrac{3}{2} N k_B$

$C_{P_0} - C_{V_0} = N k_B$ $\qquad$ $C_{P_0} = \dfrac{5}{2} N k_B$

4) $E = \underbrace{\dfrac{p_x^2}{2m} + \dfrac{p_y^2}{2m} + \dfrac{p_z^2}{2m}}_{\text{cinétique}} + \underbrace{m g_0 z}_{\text{potentielle}}$

5) $z_p = \dfrac{1}{h^3} \iiiiiint_{-\infty}^{\infty} d^3\vec{r}\, d^3\vec{p}\; e^{-\beta E(\vec{p},\vec{r})}$

$z_p = \dfrac{1}{h^3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} e^{-\beta m g_0 z}\, dz \underbrace{\left(\int_{-\infty}^{\infty} e^{-\beta \frac{p_x^2}{2m}}\, dp_x\right)^3}$

$z_p = A \dfrac{z_0}{V} \int_0^{\infty} e^{-\beta m g_0 z}\, dz$

$z_p = z_0\, z_{int}$ $\qquad$ avec $z_{int} = \dfrac{A}{V} \int_0^{L} e^{-\beta m g_0 z}\, dz$

$z_{int} = -\dfrac{A}{V} \dfrac{1}{\beta m g_0} \left[e^{-\beta m g_0 z}\right]_0^{L} = \dfrac{A}{V} \dfrac{k_B T}{m g_0} \left(1 - e^{-\beta m g_0 L}\right)$

$z_{int} = \dfrac{k_B T}{m g_0 L} \left(1 - e^{-\beta m g_0 L}\right) = \dfrac{1}{x} \left(1 - e^{-x}\right)$

$Z_p = \dfrac{z_p^N}{N!} = \dfrac{z_0^N}{N!} z_{int}^N = Z_0\, Z_{int} = Z_0\, x^{-N} \left(1 - e^{-x}\right)^N$

6) $\langle E \rangle = -\dfrac{\partial \ln Z}{\partial \beta} = -\dfrac{\partial}{\partial \beta}(\ln Z_0) - \dfrac{\partial}{\partial \beta} \ln Z_{int}$

$= \langle E_0 \rangle - \dfrac{\partial}{\partial x}(\ln Z_{int}) \times \dfrac{\partial x}{\partial \beta}$

$$\langle E \rangle = \langle E_0 \rangle - m g_0 L \frac{\partial}{\partial x} \left\{ \ln(x^{-N}) - \ln(1-e^{-x})^N \right\}$$

$$\langle E \rangle = \langle E_0 \rangle + N m g_0 L \left\{ \frac{\partial}{\partial x} \ln x + \frac{\partial}{\partial x} \ln(1-e^{-x}) \right\}$$

$$\langle E \rangle = \langle E_0 \rangle + N m g_0 L \left\{ \frac{1}{x} + \frac{-e^{-x}}{(1-e^{-x})} \right\}$$

$x \ll 1$ $\qquad m g_0 L \ll k_B T$

$e^{-x} \underset{\sim 1}{\sim} 1 - x$ et $1 - e^{-x} \sim x$

$$\langle E \rangle \simeq \langle E_0 \rangle + N m g_0 L \left\{ \frac{1}{x} - \frac{(1-x)}{x} \right\} \simeq \langle E_0 \rangle$$

$m g_0 L$ énergie potentielle négligeable
si faible champ de pesanteur
on retrouve $\langle E_0 \rangle$ gaz parfait à l'absence de champ

$x \gg 1$ $\qquad m g_0 L \gg k_B T$
fort champ de pesanteur

$e^{-x} \to 0$ $\qquad \langle E \rangle \simeq \langle E_0 \rangle + \underline{N m g_0 L}$

$$\langle E \rangle \simeq \langle E_0 \rangle + N k_B \overset{x}{T} = \frac{5}{2} N k_B T$$

$$E = \frac{p^2}{2m} + m g_0 z$$ terme non quadratique en z

$\frac{5}{2} N k_B T$ n'est pas accessible par l'application du théorème d'équipartition de l'énergie

7) $$c_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_V = \left(\frac{\partial \langle E_0 \rangle}{\partial T}\right)_V + \left(\frac{\partial \langle E_{int} \rangle}{\partial T}\right)_V$$

$$= c_{V_0} + c_{V int} = \frac{3}{2} N k_B + c_{V int}$$

$$C_{V int} = \left(\frac{\partial \langle E_{int}\rangle}{\partial x}\right)_V \frac{\partial x}{\partial T} = -\frac{mg_0L}{k_B T^2}\left(\frac{\partial\langle E_{int}\rangle}{\partial x}\right)_V$$

$$C_{V int} = -\frac{mg_0L}{k_B T^2}\frac{\partial}{\partial x}\left\{ N mg_0 L\left(\frac{1}{x} - \frac{e^{x}}{1-e^{-x}}\right)\right\}$$

$$C_{V int} = -N\left(\frac{mg_0L}{k_BT}\right)^2 k_B \left\{\frac{\partial}{\partial x}\left(\frac{1}{x}\right) - \frac{\partial}{\partial x}\left(\frac{1}{e^{x}-1}\right)\right\}$$

$$C_{V int} = -Nk_B\, x^2 \left\{-\frac{1}{x^2} + \frac{e^{x}}{(e^{x}-1)^2}\right\}$$

$$C_{V int} = +Nk_B\left\{1 - \frac{e^{x}x^2}{(e^{x}-1)^2}\right\}$$

$$C_V = C_{V0} + C_{V int} = \frac{3}{2}Nk_B + Nk_B\left\{1 - \frac{e^{x}x^2}{(e^{x}-1)^2}\right\}$$

$x \ll 1$

$$e^{x} \sim 1 + x \sim 1$$
$$e^{x} - 1 \sim x$$

$$C_V \simeq \frac{3}{2}Nk_B + Nk_B\left\{1 - \frac{x^2}{x^2}\right\} \simeq \frac{3}{2}Nk_B = C_{V0}$$

$x \gg 1$

$$C_V \simeq \frac{3}{2}Nk_B + Nk_B\left\{1 - \frac{e^{x}x^2}{e^{2x}}\right\}$$

$$C_V \simeq \frac{3}{2}Nk_B + Nk_B\left\{1 - x^2 e^{-x}\right\}$$

$$C_V \simeq \frac{3}{2}Nk_B + Nk_B = \frac{5}{2}Nk_B$$

8)

$$dP(\vec{r},\vec{p}) = f(\vec{r},\vec{p})\, d^3\vec{r}\, d^3\vec{p}$$
$$= C\, e^{-\beta E(\vec{p},\vec{r})} d^3\vec{r}\, d^3\vec{p}$$

$$\int_{-\infty}^{\infty} dP(\vec{r}\,\vec{p}) = 1 = C\int_{-\infty}^{\infty} e^{-\beta E(\vec{p},\vec{r})} d^3\vec{r}\, d^3\vec{p}$$

$$\Rightarrow C = \frac{1}{\int_{-\infty}^{\infty} e^{-\beta E(\vec{p},\vec{r})} d^3\vec{r}\, d^3\vec{p}} = \frac{1}{Z_p h^3}$$

$$dP(\vec{r},\vec{p}) = \frac{1}{z_p h^3} e^{-\beta E(\vec{p},\vec{r})} d^3\vec{p}\, d^3\vec{r} \Rightarrow$$

$$f(\vec{r},\vec{p}) = \frac{1}{z_p} \exp\left\{-\beta \frac{p^2}{2m}\right\} e^{-\beta m g_0 z}$$

$$dP(\vec{r},\vec{p}) = \frac{d^6N}{N} = \frac{1}{z_p h^3} e^{-\beta \frac{p^2}{2m}} e^{-\beta m g_0 z} dp_x\, dp_y\, dp_z\, dx\, dy\, dz$$

$$dN_z = \frac{N e^{-\beta m g_0 z}}{z_p} dz\, A \left(\int_{-\infty}^{\infty} e^{-\beta \frac{p_x^2}{2m}} dp_x \right)^3$$

$$dN_z = \frac{N A e^{-\beta m g_0 z}}{z_p} dz \left(\sqrt{\pi 2m k_B T}\right)^3$$

$$dN_z = \frac{N A e^{-\beta m g_0 z} dz}{\cancel{z_p h^3}} \frac{z_0 h^3}{V} = \frac{1}{1}$$

$$dN_z = \frac{N}{L} e^{-\beta m g_0 z} dz \frac{\cancel{z_0}}{\cancel{z_0} z_{int}} = \frac{N}{\cancel{L}} e^{-\beta m g_0 z} dz \frac{m g_0 \cancel{L}}{k_B T}$$

$$dN_z = N \frac{m g_0}{k_B T} e^{-\beta m g_0 z} dz$$

$$dN_z(z=0) = \frac{N m g_0}{k_B T} \qquad dN_z = dN_z(z=0)\, e^{-\beta m g_0 z} dz$$

$$dn_z = \frac{dN_z}{dV} = \frac{N}{A} \frac{m g_0}{k_B T} e^{-\beta m g_0 z} \frac{dz}{dz}$$

10)

$$n_z = \frac{N m g_0}{A} \beta e^{-\beta m g_0 z} \qquad P = \frac{N}{V} k_B T = n k_B T$$

$$n(z) = \frac{N m g_0}{A} \beta e^{-\beta m g_0 z} = \frac{P_0}{k_B T} e^{-\beta m g_0 z} \qquad P_0 = \frac{N m g_0}{A}$$

$$P(z) = P_0 e^{-\beta m g_0 z}$$

11) $T_r \ll T$ et $T_v \gg T$

$$\varepsilon_r = \underbrace{\frac{p_\theta^2}{2I}}_{\frac{1}{2}k_BT} + \underbrace{\frac{p_\varphi^2}{2I\sin^2\theta}}_{\frac{1}{2}k_BT} \simeq k_BT$$

$$\langle E\rangle = \frac{5}{2}Nk_BT + Nk_BT = \frac{7}{2}Nk_BT$$

$$C_V = \frac{7}{2}Nk_B$$

vibrations gelées car $T_v \gg T$

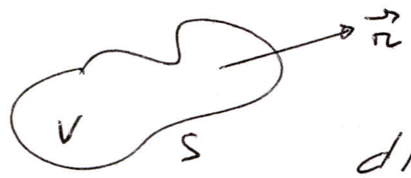
pas de contribution des modes de vibration à C_V

II stabilité d'un réacteur nucléaire

1) $$\underbrace{\vec{J}_n(\vec{r},t)}_{\substack{\text{vecteur densité}\\ \text{de flux de particules}\\ m^{-2}s^{-1}}} = -\underbrace{D}_{\text{coefficient de diffusion}}\ \overrightarrow{\text{grad}}\ \underbrace{n(\vec{r},t)}_{\substack{\text{nb de particules / volume en } \vec{r} \text{ et à } t\\ m^{-3}}}$$

$$D \equiv \frac{m^{-2}s^{-1}}{m^{-3}\,m^{-1}} = m^2s^{-1}$$

2)



$dN = \delta N^r + \delta N^p$ entre t et $t+dt$

$$dN = dt\iint_S \vec{J}_n\cdot(-\vec{n})\,dS + dt\iiint_V \sigma\, dV$$

$$\frac{dN}{dt} = \iiint_V \sigma\, dv - \iint_S \vec{J}_n\cdot\vec{n}\, dS$$

$$N = \iiint_V n\, dv \qquad \frac{dN}{dt} = \frac{d}{dt}\iiint_V n\, dV = \iiint_V \frac{\partial n}{\partial t}\, dv$$

$$\iiint_V \frac{\partial n}{\partial t} dV = \iiint_V \sigma \, dV - \iint_S \vec{J_n} \cdot \vec{n} \, dS$$

3) $$\iiint_V \frac{\partial n}{\partial t} dV = \iiint_V \sigma \, dV - \iiint \mathrm{div} \, \vec{J_n} \, dV$$

$$\boxed{\frac{\partial n}{\partial t} = \sigma - \mathrm{div} \, \vec{J_n}}$$

$$\frac{\partial n}{\partial t} = \sigma - D \, \mathrm{div} \, \vec{\mathrm{grad}} (n) = \sigma + D \cdot \Delta n$$

4) $n v \, dt$ = distance parcourue pendant dt

$\frac{n v \, dt}{L}$ = nb de neutrons absorbés pendant dt

nb de neutrons absorbés par $s = \frac{n}{\tau}$

5) nb de neutrons produits par fusion $\frac{n}{\tau} \times K$

$$\sigma = -\frac{n}{\tau} + \frac{n}{\tau} K = \frac{n}{\tau}(K-1)$$

$$\frac{\partial n}{\partial t} = D \Delta n + n \frac{(K-1)}{\tau}$$

6) $$\frac{d^2 n}{dx^2} + \frac{(K-1)}{D\tau} n = 0$$

$$\frac{d^2 n}{dx^2} + \omega^2 n = 0 \longrightarrow r^2 + \omega^2 = 0$$

$$r = \pm i\omega = \pm i \sqrt{\frac{(K-1)}{D\tau}}$$

$\Delta < 0$ si $\frac{(\cancel{K} -1)}{D\tau} > 0$; $K > 1$

$$\Delta = -4\omega^2$$

$$\sqrt{\Delta} = \pm 2i\omega$$

$$\Delta = -4 \frac{(K-1)}{D\tau}$$

$$n(x) = A \cos(\omega x) + B \sin(\omega x)$$

$\Delta > 0$ si $\frac{(K-1)}{D\tau} < 0$; $K < 1$ $\quad n(x) = A e^{\omega x} + B e^{-\omega x}$

$$\begin{cases} n(a/2) = 0 = A e^{+wa/2} + B e^{-wa/2} \\ n(-a/2) = 0 = A e^{-wa/2} + B e^{wa/2} \end{cases}$$

si A et $B \neq 0 \Rightarrow \quad A\left(e^{wa/2} - e^{-wa/2}\right) + B\left(e^{wa/2} - e^{-wa/2}\right) = 0$

$$e^{wa/2} - e^{-wa/2} = 0 \quad \underline{\text{impossible}}$$

donc $K > 1$

8)

$$n(x) = +n(-x)$$

$$A\cos(wx) + B\sin(wx) = A\cos(wx) - B\sin(wx)$$

$$B\sin(wx) = 0 \quad \forall x \Rightarrow \quad B = 0$$

$$n(x) = A\cos(wx) = A\cos\left(\sqrt{\frac{K-1}{D\tau}}\, x\right)$$

9)

$$n(a/2) = 0 = A\cos\left(w\frac{a}{2}\right) \Rightarrow \cos(wa/2) = 0$$

$$\frac{wa}{2} = (2p+1)\frac{\pi}{2} \qquad w = (2p+1)\frac{\pi}{a}$$

$$\boxed{\sqrt{\frac{K-1}{D\tau}} = (2p+1)\frac{\pi}{a}}$$