Session Janvier 2007

Oscillateur harmonique

PARTIE A] $|\psi(0)\rangle = \sum_n C_n |\varphi_n\rangle$; $H|\varphi_n\rangle = (n+\frac{1}{2})\hbar\omega|\varphi_n\rangle$

1) $H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$

$$\Rightarrow |\psi(t)\rangle = \sum_n C_n(0)\, e^{-i(n+1/2)\omega t} |\varphi_n\rangle$$

2) P : $E > 2\hbar\omega$

P_n d'avoir E_n : $P_n = |\langle\varphi_n|\psi(t)\rangle|^2 = |C_n|^2$

$\begin{cases} E_0 = \frac{\hbar\omega}{2} \\ E_1 = \frac{3\hbar\omega}{2} \\ E_2 = \frac{5\hbar\omega}{2} \end{cases}$

$\Rightarrow P = \sum_{n=2}^{\infty} |C_n|^2 = 1 - |C_0|^2 - |C_1|^2$

$P = 0 \Rightarrow |C_0|^2 + |C_1|^2 = 1 \Rightarrow C_0$ et C_1 non nuls.

3) $|\psi(0)\rangle = C_0|\varphi_0\rangle + C_1|\varphi_1\rangle \Rightarrow$ cd^t normalisation $|C_0|^2 + |C_1|^2 = 1$

$$\langle H\rangle_{\psi(0)} = P_0 E_0 + P_1 E_1 = |C_0|^2 \frac{\hbar\omega}{2} + |C_1|^2 \frac{3}{2}\hbar\omega = \hbar\omega\left(\frac{1}{2}|C_0|^2 + \frac{3}{2}|C_1|^2\right)$$
$$= \hbar\omega$$
$$\Rightarrow \frac{1}{2}|C_0|^2 + \frac{3}{2}|C_1|^2 = 1$$

ce qui donne $\begin{cases} |C_0|^2 + |C_1|^2 = 1 \\ \frac{1}{2}|C_0|^2 + \frac{3}{2}|C_1|^2 = 1 \end{cases}$ soit $|C_0|^2 = 1/2$ et $|C_1|^2 = 1/2$

4) $C_0 = \frac{1}{\sqrt{2}}$ et $|C_1| = \frac{1}{\sqrt{2}}$ $\left(C_1 = |C_1| e^{i\theta_1}\right)$

$\langle H\rangle = \hbar\omega$; $\langle X\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$ avec $X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$

- $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\varphi_0\rangle + \frac{1}{\sqrt{2}} e^{i\theta_1}|\varphi_1\rangle$

- $\langle X\rangle = \langle\psi(0)|X|\psi(0)\rangle$

$$= \left(\frac{1}{\sqrt{2}}\langle\varphi_0| + \frac{1}{\sqrt{2}} e^{-i\theta_1}\langle\varphi_1|\right)\sqrt{\frac{\hbar}{2m\omega}}(a+a^+)\left(\frac{1}{\sqrt{2}}|\varphi_0\rangle + \frac{1}{\sqrt{2}} e^{i\theta_1}|\varphi_1\rangle\right)$$

$$a|\varphi_0\rangle = 0 \; ; \; a|\varphi_1\rangle = |\varphi_0\rangle$$
$$a^+|\varphi_0\rangle = |\varphi_1\rangle \; ; \; a^+|\varphi_1\rangle = \sqrt{2}|\varphi_2\rangle$$

soit $(a+a^+)\left(\frac{1}{\sqrt{2}}|\varphi_0\rangle + \frac{1}{\sqrt{2}}e^{i\theta_1}|\varphi_1\rangle\right) = \frac{1}{\sqrt{2}}e^{i\theta_1}|\varphi_0\rangle + \frac{1}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}e^{i\theta_1}\sqrt{2}|\varphi_2\rangle$

soit $\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{2}}\langle\varphi_0| + \frac{1}{\sqrt{2}}e^{-i\theta_1}\langle\varphi_1|\right)\left(\frac{1}{\sqrt{2}}e^{i\theta_1}|\varphi_0\rangle + \frac{1}{\sqrt{2}}|\varphi_1\rangle + e^{i\theta_1}|\varphi_2\rangle\right)$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{2}e^{i\theta_1} + \frac{1}{2}e^{-i\theta_1}\right) = \sqrt{\frac{\hbar}{2m\omega}}\cos\theta_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{2m\omega}}$$

soit $\cos\theta_1 = \frac{\sqrt{2}}{2} \Rightarrow \boxed{\theta_1 = \pi/4}$

5) $|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}|\varphi_0\rangle + \frac{1}{\sqrt{2}}e^{i\pi/4}e^{-\frac{3i\omega t}{2}}|\varphi_1\rangle$

soit $\theta_1(t) = \frac{\pi}{4} - \frac{3}{2}\omega t - \frac{1}{2}\omega t = \pi/4 - \omega t$

$$\Rightarrow \langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \pi/4)$$

Partie B] $\quad e^R e^S = e^{R+S} e^{\frac{1}{2}[R,S]}$

1) $D(\alpha) = e^{\alpha a^+ - \alpha^* a} \quad \Rightarrow D^+(\alpha) = e^{\alpha^* a - \alpha a^+}$

$D^+(\alpha)D(\alpha) = D(\alpha)D^+(\alpha) = 1 \Rightarrow$ opérateur unitaire.

2) $D(\alpha)|\varphi_0\rangle \qquad R = \alpha a^+$ et $S = -\alpha^* a$

$\Rightarrow [R,S] = [\alpha a^+, -\alpha^* a] = |\alpha|^2$ car $[a,a^+] = 1$

$$D(\alpha)|\varphi_0\rangle = e^{\alpha a^+ - \alpha^* a}|\varphi_0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^+} e^{-\alpha^* a}|\varphi_0\rangle$$

calcul de $e^{-\alpha^* a}|\varphi_0\rangle = |\varphi_0\rangle - \cancel{\alpha^* a|\varphi_0\rangle} + \cancel{\frac{\alpha^{*2}}{2}a^2|\varphi_0\rangle} + \dots$

Réinjecté $e^{\alpha a^+}\left(e^{-\alpha^* a}|\varphi_0\rangle\right) = e^{\alpha a^+}|\varphi_0\rangle = \sum_n \frac{(\alpha a^+)^n}{n!}|\varphi_0\rangle$

avec $(a^+)^n|\varphi_0\rangle = \sqrt{n!}\,|\varphi_n\rangle$

Finalement $D(\alpha)|\varphi_0\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \alpha^n \frac{\sqrt{n!}}{n!}|\varphi_n\rangle = \sum_n \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{n!}}\alpha^n|\varphi_n\rangle = |\alpha\rangle$ cqfd.

3) Calcul $\left[a, a^{+2}\right] = \left[a, a^{+}a^{+}\right] = a a^{+} a^{+} - a^{+} a^{+} a \left(- a^{+} a a^{+} + a^{+} a a^{+}\right)$

$$= \left[a, a^{+}\right] a^{+} + a^{+}\left[a, a^{+}\right] = 2\left[a, a^{+}\right] a^{+}$$

$\left[a, a^{+n}\right] = n\left[a, a^{+}\right] a^{+(n-1)}$ admis

calcul $\left[a, a^{+(n+1)}\right] = \left[a, a^{+} a^{+n}\right] = a a^{+} a^{+n} - a^{+} a^{+n} a \left(- a^{+} a a^{+n} + a^{+} a a^{+n}\right)$

$$= \left[a, a^{+}\right] a^{+n} + a^{+}\left[a, a^{+n}\right]$$

$$= \left[a, a^{+}\right] a^{+n} + a^{+} n \left[a, a^{+}\right] a^{+(n-1)}$$

$$= (n+1)\left[a, a^{+}\right] a^{+n}$$ démontré à l'ordre $n+1$

on considère $F(a^{+}) = \sum_n C_n a^{+n}$

calcul $\left[a, F(a^{+})\right] = \sum_n C_n \left[a, a^{+n}\right] = \sum_n C_n\, n \underbrace{\left[a, a^{+}\right]}_{=1} a^{+(n-1)}$

$$= \sum_n C_n\, n\, a^{+(n-1)}$$

et $\dfrac{dF(a^{+})}{da^{+}} = \dfrac{d}{da^{+}} \sum_n C_n a^{+n} = \sum_n C_n\, n\, a^{+(n-1)}$

$\Rightarrow \left[a, F(a^{+})\right] = \dfrac{d}{da^{+}} F(a^{+})$

Reprise des calculs avec $G(a) = \sum_n B_n a^{n}$; $\left[a^{+}, a^{n}\right] = n\left[a^{+}, a\right] a^{n-1}$

Calcul $\left[a^{+}, G(a)\right] = \sum_n B_n \left[a^{+}, a^{n}\right] = -\sum_n B_n\, n\, a^{n-1}$

et $\dfrac{dG(a)}{da} = \sum_n B_n\, n\, a^{n-1}$

$\Rightarrow \left[a^{+}, G(a)\right] = -\dfrac{d}{da} G(a)$

$\left[a, D(\alpha)\right]|\varphi_0\rangle = a D(\alpha)|\varphi_0\rangle - \cancel{D(\alpha) a|\varphi_0\rangle} = a|\alpha\rangle = \alpha|\alpha\rangle$
$= \alpha D(\alpha)|\varphi_0\rangle$

Comparaison $\left[a, D(\alpha)\right] / \left[a, F(a^{+})\right] \Rightarrow \left[a, D(\alpha)\right] = \left[a, F(a^{+})\right] = \alpha D(\alpha)$

<u>Calcul $D^{-1}(\alpha)\, a\, D(\alpha)$</u> : $\left[a, D(\alpha)\right] = a D(\alpha) - D(\alpha) a \Leftrightarrow a D(\alpha) = \left[a, D(\alpha)\right] + D(\alpha) a$

$$D^{-1}(\alpha)\left(a D(\alpha)\right) = D^{-1}(\alpha)\left(\left[a, D(\alpha)\right]\right) + D^{-1}(\alpha) D(\alpha) a.$$

$$= D^{-1}(\alpha)\, \alpha\, D(\alpha) + a = \alpha + a = D^{-1}(\alpha)\, a\, D(\alpha)$$

<u>Calcul $D^{-1}_{(\alpha)} a^{+} D_{(\alpha)}$</u> : $\left(D^{-1}(\alpha)\, a\, D(\alpha)\right)^{+} = \alpha^{*} + a^{+} = D^{+}(\alpha)\, a^{+} D^{-1+}_{(\alpha)} = D^{-1}(\alpha)\, a^{+} D(\alpha) = \alpha^{*} + a^{+}$

4) Dem : $a\, D^{-1}(\alpha)\, |\alpha\rangle = 0$

$$\underbrace{D^{-1}(\alpha)\, a\, D(\alpha)}\left(D^{-1}(\alpha)\,|\alpha\rangle\right) = D^{-1}(\alpha)\, a\, |\alpha\rangle = \alpha\, D^{-1}(\alpha)\, |\alpha\rangle$$

$$= (\alpha + a)\, D^{-1}(\alpha)\,|\alpha\rangle = \alpha\, D^{-1}(\alpha)\,|\alpha\rangle + a\, D^{-1}(\alpha)\,|\alpha\rangle$$

donc $a\, \underbrace{D^{-1}(\alpha)\,|\alpha\rangle} = 0$

$$D^{-1}(\alpha)\,|\alpha\rangle = |0\rangle$$

et donc $D(\alpha)\, D^{-1}(\alpha)\, |\alpha\rangle = |\alpha\rangle = D(\alpha)\,|0\rangle$

cqfd.

Operateur d'évolution d'un spin 1/2 $\qquad \vec{M} = \gamma \vec{S}$

$$\vec{B}_0 \begin{vmatrix} -\dfrac{\omega_x}{\gamma} \\ -\dfrac{\omega_y}{\gamma} \\ -\dfrac{\omega_z}{\gamma} \end{vmatrix}$$

1) $U(t) = e^{-\frac{iWt}{\hbar}}$ avec $W = -\vec{M}\cdot\vec{B}_0 = -\gamma\, \vec{S}\cdot\vec{B}_0$

$$= -\gamma\left(S_x B_x + S_y B_y + S_z B_z\right)$$

$$= \left(\omega_x S_x + \omega_y S_y + \omega_z S_z\right)$$

soit $\Omega = \frac{1}{\hbar}\left(\omega_x S_x + \omega_y S_y + \omega_z S_z\right)$

$$\Omega = \frac{1}{2}\left[\omega_x \sigma_x + \omega_y \sigma_y + \omega_z \sigma_z\right] = \frac{1}{2}\left[\begin{pmatrix} 0 & \omega_x \\ \omega_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\omega_y \\ i\omega_y & 0 \end{pmatrix} + \begin{pmatrix} \omega_z & 0 \\ 0 & -\omega_z \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} \omega_z & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_z \end{pmatrix}$$

$$\Omega^2 = \frac{1}{4}\begin{pmatrix} \omega_z & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_z \end{pmatrix}\begin{pmatrix} \omega_z & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \omega_z^2 + \omega_x^2 + \omega_y^2 & 0 \\ 0 & \omega_x^2 + \omega_y^2 + \omega_z^2 \end{pmatrix}$$

$$= \frac{1}{4}\left(\omega_x^2 + \omega_y^2 + \omega_z^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{\omega_0}{2}\right)^2$$

2) $U(t) = e^{-i\Pi t} = \cos(\Pi t) - i \sin(\Pi t)$

$$\cos(\Pi t) = \sum_n \frac{(-1)^n (\Pi t)^{2n}}{(2n)!} = \sum_n (-1)^n \left(\frac{\omega_0}{2}\right)^{2n} \frac{t^{2n}}{(2n)!} = \cos\left(\frac{\omega_0 t}{2}\right)$$

$$\Pi \sin(\Pi t) = \Pi \sum_n (-1)^n \frac{(\Pi t)^{2n+1}}{(2n+1)!} = \sum_n (-1)^n \frac{(\Pi t)^{2n+2}}{t\,(2n+1)!} = \sum_n (-1)^n \frac{(\Pi t)^{2(n+1)}}{t\,(2n+1)!}$$

$$= \sum_n \frac{(-1)^n\, t^{2(n+1)}}{t\,(2n+1)!} \left(\frac{\omega_0}{2}\right)^{2(n+1)} = \frac{\omega_0}{2} \sum_n \frac{(-1)^n \left(\frac{\omega_0 t}{2}\right)^{2n+1}}{(2n+1)!} = \frac{\omega_0}{2} \sin\left(\frac{\omega_0 t}{2}\right)$$

donc $U(t) = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i\omega_0}{2\Pi} \sin\left(\frac{\omega_0 t}{2}\right) = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i\omega_0 \Pi}{2\underbrace{\Pi^2}_{=\omega_0^2/4}} \sin\left(\frac{\omega_0 t}{2}\right)$

$$= \cos\left(\frac{\omega_0 t}{2}\right) - \frac{2i\Pi}{\omega_0} \sin\left(\frac{\omega_0 t}{2}\right) \quad \text{cqfd.}$$

<u>Matrice U(t)</u>

$$U(t) = \cos\left(\frac{\omega_0 t}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{\omega_0} \sin\left(\frac{\omega_0 t}{2}\right) \begin{pmatrix} \omega_3 & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_3 \end{pmatrix}$$

$$U(t) = \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 & \frac{-i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)(\omega_x - i\omega_y) \\ \frac{-i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)(\omega_x + i\omega_y) & \cos\left(\frac{\omega_0 t}{2}\right) + \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

3) $|\psi(0)\rangle = |+\rangle \quad ; \quad |\psi(t)\rangle = U(t)|+\rangle = \alpha|+\rangle + \beta|-\rangle$

$$\mathcal{P}_{++}(t) = |\langle + | \psi(t)\rangle|^2 = |\langle + | U(t) | + \rangle|^2 \quad \text{cqfd.}$$

calcul de $U(t)|+\rangle = \begin{pmatrix} \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3 \\ \frac{-i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)(\omega_x + i\omega_y) \end{pmatrix}$ (1,0)

$$\langle + | U(t) | + \rangle = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{i}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\omega_3.$$

$$\Rightarrow \quad P_{++}(t) = |\langle + | U(t) | + \rangle|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right) + \sin^2\left(\frac{\omega_0 t}{2}\right)\left(\frac{\omega_3}{\omega_0}\right)^2$$

$$= \left(1 - \sin^2\left(\frac{\omega_0 t}{2}\right)\right) + \left(\frac{\omega_3}{\omega_0}\right)^2 \sin^2\left(\frac{\omega_0 t}{2}\right)$$

$$= 1 - \frac{(\omega_x^2 + \omega_y^2)}{\omega_0^2} \sin^2\left(\frac{\omega_0 t}{2}\right)$$

$$P_{max} = 1 - \frac{(\omega_x^2 + \omega_y^2)}{\omega_0^2}$$