Système de spins (8 points)

1) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$

$\hat{H}(t) = \omega_0(t)\,\hat{S}_z$ avec $\omega_0(t) = \omega_0 \frac{t}{T}$

$\underbrace{\hat{H}}_{\omega_0(t)\hat{S}_z}|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle$ on cherche $|\psi(t)\rangle = c_+(t)|+\rangle + c_-|-\rangle$

soit $\omega_0(t)\hat{S}_z|\psi(t)\rangle = \omega_0(t)\left(c_+(t)\frac{\cancel{\hbar}}{2}|+\rangle - c_-(t)\frac{\cancel{\hbar}}{2}|-\rangle\right) = i\cancel{\hbar}(c_+'(t)|+\rangle + c_-'(t)|-\rangle)$

1pt on assimile
$$\left.\begin{aligned} \frac{\omega_0(t)\,c_+(t)}{2} &= i\frac{dc_+(t)}{dt} \\ -\frac{\omega_0(t)\,c_-(t)}{2} &= i\frac{dc_-(t)}{dt} \end{aligned}\right\} \text{2 eqs diff découplées}$$

1pt SOLUTIONS
$$c_+(t) = \underbrace{c_+(0)}_{=1/\sqrt{2}\ \to\ \text{cdt initiale}}\, e^{-\frac{i\omega_0 t^2}{4T}}$$
$$c_-(t) = \underbrace{c_-(0)}_{=\frac{i}{\sqrt{2}}\ \to\ \text{cdt initiale}}\, e^{\frac{i\omega_0 t^2}{4T}}$$

à réinjecter dans $\frac{dc_+}{dt} + \frac{i\omega_0 t}{2T}c_+ = 0$; $\frac{dc_-}{dt} - \frac{i\omega_0 t}{2T}c_- = 0$

1pt soit $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i\omega_0 t^2}{4T}}|+\rangle + i\,e^{+\frac{i\omega_0 t^2}{4T}}|-\rangle\right]$

1pt d'où $\boxed{\theta(t) = -\frac{\omega_0 t^2}{4T}}$

2) $t = \tau$ mesure de $\hat{S}_y$ pour $t = \tau > T$ plus de perturbation
$\Rightarrow \hat{H}(t) = 0 \Rightarrow |\psi(t)\rangle = |\psi(T)\rangle$

1pt 2 possibilités $\begin{cases} +\frac{\hbar}{2} \\ -\frac{\hbar}{2} \end{cases}$ $\mathcal{P}(+\frac{\hbar}{2}) = |\langle S_y^+|\psi(T)\rangle|^2$

avec $\langle S_y^+|\psi(T)\rangle = \left(\frac{1}{\sqrt{2}} ; \frac{-i}{\sqrt{2}}\right)\begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\frac{i\omega_0 T}{4}} \\ \frac{i}{\sqrt{2}}e^{\frac{i\omega_0 T}{4}} \end{pmatrix}$
$= \frac{e^{-\frac{i\omega_0 T}{4}} + e^{\frac{i\omega_0 T}{4}}}{2} = \cos\left(\frac{\omega_0 T}{4}\right)$

1pt $\Rightarrow \mathcal{P}(+\frac{\hbar}{2}) = \cos^2\left(\frac{\omega_0 T}{4}\right)$ et $\mathcal{P}(-\frac{\hbar}{2}) = \sin^2\left(\frac{\omega_0 T}{4}\right) = 1 - \mathcal{P}(+\frac{\hbar}{2})$

1pt

Être sûr du résultat $\Rightarrow \dfrac{\omega_0 T}{4} = n\pi \quad \left(P(+\tfrac{\hbar}{2}) = 1 \text{ et } P(-\tfrac{\hbar}{2}) = 0\right)$

$\Leftrightarrow T = \dfrac{4n\pi}{\omega_0}$

$\Rightarrow \dfrac{\omega_0 T}{4} = (2n+1)\dfrac{\pi}{2} \quad \left(P(+\tfrac{\hbar}{2}) = 0 \text{ et } P(-\tfrac{\hbar}{2}) = 1\right)$

$\Leftrightarrow T = \dfrac{2\pi(2n+1)}{\omega_0}$

1pt

Phénomène de résonance.

Oscillateur harmonique chargé

PARTIE A ⇒ 9 pts

1) $|\alpha\rangle = \sum_n C_n |n\rangle$ base $|n\rangle$

récurrence $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{H}|n\rangle = E_n |n\rangle$

$$\begin{cases} \hat{a}|\alpha\rangle = \hat{a}\sum_n C_n |n\rangle = \sum_n C_n \hat{a}|n\rangle = \sum_n C_n \sqrt{n}\,|n-1\rangle \\ \hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \alpha \sum_n C_n |n\rangle \end{cases}$$

Multiplions par le bras $\langle m|$: $\alpha \sum_n C_n \underbrace{\langle m|n\rangle}_{\substack{\delta_{mn} \\ \delta=1\ m=n}} = \sum_n C_n \sqrt{n} \underbrace{\langle m|n-1\rangle}_{\substack{\delta_{m\,n-1} \\ \delta=1\ (m=n-1) \\ \text{ou } n=m+1}}$

$$\alpha C_m = C_{m+1}\sqrt{m+1}$$

soit $C_{m+1} = \dfrac{\alpha C_m}{\sqrt{m+1}}$ relation de récurrence

$\left\| \text{ ou } C_n = \alpha \dfrac{C_{n-1}}{\sqrt{n}} = \dfrac{\alpha}{\sqrt{n}}\left(\dfrac{\alpha}{\sqrt{n-1}} C_{n-2}\right) \dots = \dfrac{\alpha^n}{\sqrt{n!}} C_0 \right.$ premier coefficient

2pts

2) Normalisation $\langle\alpha|\alpha\rangle = 1 = \sum_n |C_n|^2 \underbrace{\langle n|n\rangle}_{1 \text{ base orthonormée}} = \sum_n |C_n|^2 = \sum_n \dfrac{|\alpha|^{2n}}{n!} |C_0|^2$

$$= |C_0|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} = 1$$

$\Rightarrow |C_0|^2 = \dfrac{1}{e^{|\alpha|^2}} = e^{-|\alpha|^2}$

soit $|C_0| = e^{-|\alpha|^2/2}$

2pts

$\Rightarrow$ État propre $|\alpha\rangle$ de $\hat{a}$ quelque soit α

$|C_0|^2$

3) Probabilité de mesure $E_n = (n+\tfrac{1}{2})\hbar\omega$: $P_n = |C_n|^2 = \dfrac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}$

1pt

4) Valeur moyenne de l'énergie

$$\langle \hat{H} \rangle_\alpha = \langle E \rangle = \sum_n P_n E_n = \sum_n \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} (n+1/2)\hbar\omega$$

$n! = n(n-1)!$

1 pt pour 1 des 2 méthodes

$$= \sum_n \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} n\hbar\omega + \underbrace{\sum_n \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}}_{=1} \frac{\hbar\omega}{2}$$

$|\alpha|^{2n-2+2}$

$$= \sum_n \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{(n-1)!} \hbar\omega + \frac{\hbar\omega}{2}$$

$$= |\alpha|^2 \sum_n \frac{|\alpha|^{2(n-1)} e^{-|\alpha|^2}}{(n-1)!} \hbar\omega + \frac{\hbar\omega}{2}$$

$$= |\alpha|^2 \hbar\omega + \frac{\hbar\omega}{2} = \hbar\omega\left(|\alpha|^2 + 1/2\right)$$

Autre méthode $\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + 1/2)$

$$\langle \hat{H} \rangle_\alpha = \hbar\omega \langle \alpha | (\hat{a}^+\hat{a} + \tfrac{1}{2}) | \alpha \rangle$$

$$= \hbar\omega \langle \alpha | \underbrace{\hat{a}^+}_{\alpha^* \langle \alpha |} \underbrace{\hat{a} | \alpha \rangle}_{\alpha | \alpha \rangle} + \frac{\hbar\omega}{2} \underbrace{\langle \alpha | \alpha \rangle}_{=1}$$

$$= \hbar\omega\left(|\alpha|^2 + \tfrac{1}{2}\right)$$

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5) $| \alpha(t) \rangle = \sum_n c_n e^{-\frac{iE_n t}{\hbar}} | n \rangle$ avec $c_n = \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}$

2 pts

$$= \sum_n \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}} e^{-\frac{i(n+1/2)\hbar\omega t}{\hbar}} | n \rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} e^{-in\omega t} e^{-\frac{i\omega t}{2}} | n \rangle$$

$$= e^{-\frac{i\omega t}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} e^{-in\omega t} | n \rangle$$

Posons $\alpha' = \alpha e^{-i\omega t}$ $\Leftrightarrow$ $\alpha'^n = \alpha^n e^{-in\omega t}$

et $|\alpha'| = |\alpha|$

soit $| \alpha(t) \rangle = e^{-\frac{i\omega t}{2}} \underbrace{\sum_n \frac{\alpha'^n}{n!} e^{-|\alpha'|^2/2} | n \rangle}_{|\alpha'\rangle \Rightarrow \text{état propre de } \hat{a}} = e^{-\frac{i\omega t}{2}} | \alpha' \rangle$

PARTIE B ⇒ 8 points

1) $w(t) = -q\,\mathcal{E}(t)\,\hat{X}$

$$\begin{cases}\hat{X}' = \sqrt{\frac{m\omega}{\hbar}}\,\hat{X}\\ \hat{P}' = \frac{\hat{P}}{\sqrt{m\hbar\omega}}\end{cases} \qquad \begin{cases}\hat{a} = \frac{\hat{X}'+i\hat{P}'}{\sqrt{2}}\\ a^+ = \frac{\hat{X}'-i\hat{P}'}{\sqrt{2}}\end{cases}$$

on trouve $$\begin{cases}\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}+\hat{a}^+)\\ \hat{P} = -i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}-\hat{a}^+)\end{cases}$$

Hamiltonien → $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}\overbrace{m\omega^2}^{k}\hat{X}^2 - q\,\mathcal{E}(t)\,\hat{X}$

soit $$\begin{cases}\hat{X}^2 = \frac{\hbar}{2m\omega}(\hat{a}+\hat{a}^+)^2\\ \hat{P}^2 = -\frac{m\hbar\omega}{2}(\hat{a}-\hat{a}^+)^2\end{cases}$$

$$= -\frac{\hbar\omega}{4}(\hat{a}-\hat{a}^+)^2 + \frac{\hbar\omega}{4}(\hat{a}+\hat{a}^+)^2 - q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+a^+)$$

$$= -\frac{\hbar\omega}{4}(\cancel{\hat{a}^2}+\cancel{\hat{a}^{+2}}-\hat{a}\hat{a}^+-\hat{a}^+\hat{a}) + \frac{\hbar\omega}{4}(\cancel{\hat{a}^2}+\cancel{\hat{a}^{+2}}+\hat{a}\hat{a}^++\hat{a}^+\hat{a}) - q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)$$

$$\hat{H}(t) = \frac{\hbar\omega}{2}(\hat{a}^+\hat{a}+\hat{a}\hat{a}^+) - q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)$$

or $[\hat{a},\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$

2pts ou $$\boxed{\hat{H}(t) = \underbrace{\hbar\omega\left(\hat{a}\hat{a}^+-\frac{1}{2}\right)}_{H_0} \underbrace{- q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^+)}_{W(t)}}$$

commutateur → $[\hat{a},\hat{H}(t)] = [\hat{a},\hat{H}_0+w(t)] = [\hat{a},\hat{H}_0] + [\hat{a},\hat{W}(t)]$

$$= \hbar\omega\left[\hat{a},\hat{a}\hat{a}^+-\frac{1}{2}\right] - q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}[\hat{a},\hat{a}+\hat{a}^+]$$

calcul de $\left[\hat{a},\hat{a}\hat{a}^+-\frac{1}{2}\right] = \hat{a}\hat{a}\hat{a}^+ - \cancel{\frac{\hat{a}}{2}} - \hat{a}\hat{a}^+\hat{a} + \cancel{\frac{\hat{a}}{2}} = \hat{a}\underbrace{[\hat{a},\hat{a}^+]}_{=1} = \hat{a}$

$$[\hat{a},\hat{a}+\hat{a}^+] = \cancel{\hat{a}\hat{a}} + \hat{a}\hat{a}^+ - \cancel{\hat{a}\hat{a}} - \hat{a}^+\hat{a} = [\hat{a},\hat{a}^+] = 1$$

0,5pt d'où $[\hat{a},\hat{H}(t)] = \hbar\omega\hat{a} - q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}$ ⇒ $\boxed{\beta = 1/2}$

un calcul identique donne

commutateur → 0,5pt $[\hat{a}^+,\hat{H}(t)] = -\hbar\omega\hat{a}^+ + q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}$

2) $\alpha(t) = \langle\psi(t)|\hat{a}|\psi(t)\rangle$

$$\frac{d}{dt}\langle\psi(t)|\hat{a}|\psi(t)\rangle = \left(\frac{d}{dt}\langle\psi(t)|\right)\hat{a}|\psi(t)\rangle + \langle\psi(t)|\frac{d}{dt}(\hat{a}|\psi(t)\rangle)$$

calcul de $\frac{d}{dt}\langle\psi(t)|$ on a $\langle\psi(t)|\hat{H} = -i\hbar \frac{d}{dt}\langle\psi(t)|$

$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}$ soit $\left(\frac{d}{dt}\langle\psi(t)|\right)\hat{a}|\psi(t)\rangle = \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{a}|\psi(t)\rangle$

calcul de $\frac{d}{dt}\left(\hat{a}|\psi(t)\rangle\right)$ on a $\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$

et $\hat{a}\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}\left(\hat{a}|\psi(t)\rangle\right)$

$= \frac{-i}{\hbar}\hat{a}\hat{H}|\psi(t)\rangle$ soit $\langle\psi(t)|\frac{d}{dt}\left(\hat{a}|\psi(t)\rangle\right) = \frac{-i}{\hbar}\langle\psi(t)|\hat{a}\hat{H}|\psi(t)\rangle$

AU FINAL : $\frac{d\alpha(t)}{dt} = -\frac{i}{\hbar}\langle\psi(t)|[\hat{a},\hat{H}(t)]|\psi(t)\rangle$ => commutateur précédemment calculé

$\left(\text{ou } i\hbar\frac{d\alpha(t)}{dt} = \langle\psi(t)|[\hat{a},\hat{H}(t)]|\psi(t)\rangle\right)$

$$\frac{d\alpha(t)}{dt} = \frac{-i}{\hbar}\langle\psi(t)|\left(\hbar\omega\hat{a} - q\mathcal{E}(t)\sqrt{\frac{\hbar}{2m\omega}}\right)|\psi(t)\rangle$$

1,5 pt

$$= -i\omega\underbrace{\langle\psi(t)|\hat{a}|\psi(t)\rangle}_{\equiv\alpha(t)} + i\underbrace{q\mathcal{E}(t)\sqrt{\frac{1}{2m\hbar\omega}}}_{\equiv d(t)}$$

=> eq diff $\frac{d\alpha(t)}{dt} = -i\omega\,\alpha(t) + i\,d(t)$

0,5 pt <u>Intégration de l'eq diff</u> $\frac{d\alpha}{dt} + i\omega\alpha = i\,d \Rightarrow \alpha = \alpha_0 e^{-i\omega t} + f(t)$

terme de forçage (champ électrique)

<u>Valeurs moyennes des positions et impulsions</u> $\alpha = \langle\psi(t)|\hat{a}|\psi(t)\rangle$

$\alpha^* = \langle\psi(t)|\hat{a}^+|\psi(t)\rangle$

observables $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$; $\hat{P} = \sqrt{\frac{m\hbar\omega}{2}}\,i(\hat{a}^+ - \hat{a})$

1 pt $\langle\hat{X}\rangle_\psi = \sqrt{\frac{\hbar}{2m\omega}}(\alpha + \alpha^*)$; $\langle\hat{P}\rangle_\psi = i\sqrt{\frac{m\hbar\omega}{2}}(\alpha^* - \alpha)$

$\alpha = |\alpha_0|e^{i\varphi}e^{-i\omega t} + f(t)$

$\alpha^* = |\alpha_0|e^{-i\varphi}e^{i\omega t} + f^*(t)$

3) $|\varphi(t)\rangle = [\hat{a} - \alpha(t)]\,|\psi(t)\rangle$

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = i\hbar\frac{d}{dt}[\hat{a}-\alpha(t)]|\psi(t)\rangle = \underbrace{i\hbar\frac{d}{dt}\left(\hat{a}|\psi(t)\rangle\right)}_{①} - \underbrace{i\hbar\frac{d}{dt}\left(\alpha(t)|\psi(t)\rangle\right)}_{②}$$

① $\Longrightarrow$ or (voir plus haut) $i\hbar\frac{d}{dt}\left(\hat{a}|\psi(t)\rangle\right) = \hat{a}\hat{H}|\psi(t)\rangle$

② $\Longrightarrow$ $i\hbar\frac{d}{dt}\left(\alpha(t)|\psi(t)\rangle\right) = i\hbar\left(\frac{d\alpha(t)}{dt}\right)|\psi(t)\rangle + i\hbar\,\alpha(t)\frac{d}{dt}|\psi(t)\rangle$

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle - \overbrace{\hat{a}\hat{H}|\psi(t)\rangle}^{①} = -i\hbar\left(-i\omega\alpha(t) + i\dot{\alpha}(t)\right)|\psi(t)\rangle + \alpha(t)\hat{H}|\psi(t)\rangle$$
$$= -\hbar\omega\alpha(t)|\psi(t)\rangle + \hbar\dot{\alpha}(t)|\psi(t)\rangle + \alpha(t)\hat{H}|\psi(t)\rangle$$

1 pt

$$i\hbar\frac{d}{dt}|\varphi(t)\rangle = \underbrace{[\hat{a},\hat{H}]|\psi(t)\rangle + \hat{H}\hat{a}|\psi(t)\rangle}_{\hat{a}\hat{H}|\psi(t)\rangle} - \hbar\omega\alpha(t)|\psi(t)\rangle + \hbar\dot{\alpha}(t)|\psi(t)\rangle + \alpha(t)\hat{H}|\psi(t)\rangle$$
$$= \hbar\omega\hat{a}|\psi(t)\rangle - \cancel{\hbar\dot{\alpha}(t)|\psi(t)\rangle} + \hat{H}\hat{a}|\psi(t)\rangle - \hbar\omega\alpha(t)|\psi(t)\rangle + \cancel{\hbar\dot{\alpha}(t)|\psi(t)\rangle} + \alpha(t)\hat{H}|\psi(t)\rangle$$
$$= \hbar\omega\hat{a}|\psi(t)\rangle + \hat{H}\hat{a}|\psi(t)\rangle - \hbar\omega\alpha(t)|\psi(t)\rangle + \alpha(t)\hat{H}|\psi(t)\rangle$$
$$= \hbar\omega(\hat{a}-\alpha(t))|\psi(t)\rangle + \hat{H}(\hat{a}-\alpha(t))|\psi(t)\rangle$$
$$= (\hbar\omega+\hat{H})\underbrace{(\hat{a}-\alpha(t))|\psi(t)\rangle}_{|\varphi(t)\rangle} = (\hbar\omega+\hat{H})|\varphi(t)\rangle \quad \text{cqfd}$$

<u>Variation temporelle de la norme</u>

$$\frac{d}{dt}\langle\varphi(t)|\varphi(t)\rangle = \frac{d}{dt}\left(\langle\varphi(t)|\right)|\varphi(t)\rangle + \langle\varphi(t)|\frac{d}{dt}|\varphi(t)\rangle$$

1 pt

$$= \frac{i}{\hbar}(\hat{H}+\hbar\omega)\langle\varphi(t)|\varphi(t)\rangle + \langle\varphi(t)|\frac{-i}{\hbar}(\hat{H}+\hbar\omega)|\varphi(t)\rangle$$
$$= \cancel{\frac{i}{\hbar}(\hat{H}+\hbar\omega)\langle\varphi(t)|\varphi(t)\rangle} - \cancel{\frac{i}{\hbar}(\hat{H}+\hbar\omega)\langle\varphi(t)|\varphi(t)\rangle}$$
$$= 0 \quad \text{norme constante}$$