Exercice I

1/ a) $\overrightarrow{OA} = r\vec{e}_r = r\sin\theta_0\,\vec{e}_{x'} + r\cos\theta_0\,\vec{e}_z$

$\vec{v}(A/R') = \left[\dfrac{d\overrightarrow{OA}}{dt}\right]_{R'} = \dot{r}\sin\theta_0\,\vec{e}_{x'} + \dot{r}\cos\theta_0\,\vec{e}_z \Rightarrow \vec{a}(A/R') = \ddot{r}(\sin\theta_0\,\vec{e}_{x'} + \cos\theta_0\,\vec{e}_z)$

b) $\vec{v}(O' \in R'/R) = \vec{0}$ $\Rightarrow$ pas de translation.
$\vec{\Omega}(R'/R) = \omega_0\vec{e}_z$
$\Rightarrow$ Mvt de rot. pure de R'/R.

c) $\vec{v}_e(A, R'/R) = \cancel{\vec{v}(O' \in R'/R)} + \vec{\Omega}(R'/R) \times \overrightarrow{O'A}$
$= \omega_0\vec{e}_z \times (r\sin\theta_0\,\vec{e}_{x'} + r\cos\theta_0\,\vec{e}_z) = r\omega_0\sin\theta_0\,\vec{e}_{y'}$.

$\vec{a}_e(A, R'/R) = \cancel{\vec{a}(O' \in R'/R)} + \underbrace{\left[\dfrac{d\vec{\Omega}}{dt}\right]_R}_{=\vec{0}} \times \overrightarrow{O'A} + \vec{\Omega} \times (\vec{\Omega} \times \overrightarrow{O'A})$.

$= \omega_0^2\vec{e}_z \times (\vec{e}_z \times r\sin\theta_0\,\vec{e}_{x'}) = -r\omega_0^2\sin\theta_0\,\vec{e}_{x'}$

$\vec{a}_c(A, R'/R) = 2\vec{\Omega}(R'/R) \times \vec{v}(M/R')$
$= 2\omega_0\vec{e}_z \times (\dot{r}\sin\theta_0\,\vec{e}_{x'} + \dot{r}\cos\theta_0\,\vec{e}_z) = 2\omega_0\dot{r}\sin\theta_0\,\vec{e}_{y'}$.

d) $\vec{v}(A/R) = \vec{v}(A/R') + \vec{v}_e(A, R'/R)$
$= (\dot{r}\sin\theta_0\,\vec{e}_{x'} + \dot{r}\cos\theta_0\,\vec{e}_z) + r\omega_0\sin\theta_0\,\vec{e}_{y'}$

$\vec{a}(A/R) = \vec{a}(A/R') + \vec{a}_e(A, R'/R) + \vec{a}_c(A, R'/R)$
$= \ddot{r}(\sin\theta_0\,\vec{e}_{x'} + \cos\theta_0\,\vec{e}_z) - r\omega_0^2\sin\theta_0\,\vec{e}_{x'} + 2\omega_0\dot{r}\sin\theta_0\,\vec{e}_{y'}$.

Directement :

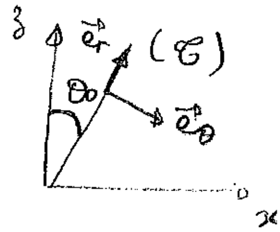
$\vec{v}(A/R) = \left[\dfrac{d\overrightarrow{OA}}{dt}\right]_R = \dot{r}\sin\theta_0\,\vec{e}_{x'} + r\sin\theta_0\underbrace{\left[\dfrac{d\vec{e}_{x'}}{dt}\right]_R}_{\omega_0\vec{e}_{y'}} + \dot{r}\cos\theta_0\,\vec{e}_z$ CQFD.

$\vec{a}(A/R) = \left[\dfrac{d\vec{v}(A/R)}{dt}\right]_R = \ddot{r}\sin\theta_0\,\vec{e}_{x'} + \dot{r}\sin\theta_0\underbrace{\left[\dfrac{d\vec{e}_{x'}}{dt}\right]_R}_{\omega_0\vec{e}_{y'}} + \dot{r}\omega_0\sin\theta_0\,\vec{e}_{y'} + r\omega_0\sin\theta_0\underbrace{\left[\dfrac{d\vec{e}_{y'}}{dt}\right]_R}_{-\omega_0\vec{e}_{x'}} + \ddot{r}\cos\theta_0\,\vec{e}_z$.

CQFD.

$\Omega(R'/R) = \omega_0 \vec{e_z}$

II ① Etude du mvt dans R', non galiléen $\Rightarrow \vec{F_i}$



a) $\vec{P} = -mg\vec{e_z}$

$\vec{T}$, $\perp$ tige car pas de frottements $\Rightarrow \vec{T} = T_\theta\vec{e_\theta} + T_{y'}\vec{e_{y'}}$

$\vec{F_{ic}} = -2m\vec{\Omega}(R'|R)\times\vec{v}(M|R') = -2m\omega_0\vec{e_z}\times\dot{r}\vec{e_r} = -2m\omega_0\dot{r}\sin\theta_0\,\vec{e_{y'}}$

$$\vec{F_{ie}} = -m\left\{\underbrace{\vec{a}(O'\in R'|R)}_{O\equiv O'\text{ fixe}} + \underbrace{\left[\frac{d\vec{\Omega}(R'|R)}{dt}\right]_R}_{=0,\text{ car }\omega_0 = cte}\times\vec{O'M} + \vec{\Omega}(R'|R)\times\left[\vec{\Omega}(R'|R)\times\vec{O'M}\right]\right\}$$

$\Rightarrow \vec{F_{ie}} = -m\omega_0^2 r\,\vec{e_z}\times(\vec{e_z}\times\vec{e_r}) = m\omega_0^2 r\cancel{\sin\theta_0}\,\vec{e_{x'}}$

b) $d\mathcal{E}_{p_{ie}} = -\delta W_{ie} = -\vec{F_{ie}}\cdot\vec{dl} = -(m\omega_0^2\sin\theta_0)\,r\,dx'$ $\quad x' = r\sin\theta_0 \Rightarrow dx' = \sin\theta_0$

$\Rightarrow \mathcal{E}_{p_{ie}}(r) = -(m\omega_0^2\sin^2\theta_0)\frac{r^2}{2} + cte \Rightarrow \mathcal{E}_{p_{ie}}(r) = -\frac{m\omega_0^2\sin^2\theta_0}{2}r^2.$

c) $\vec{T} \perp$ tige $\Rightarrow \mathcal{E}_p(\vec{T}) = 0$

$\vec{F_{ic}} \perp$ vitesse $\Rightarrow \vec{F_{ic}}$ ne travaille pas.

Donc $\mathcal{E}_p(\Sigma\vec{F}) = \mathcal{E}_p(\vec{P} + \vec{F_{ie}}) = +mgr\cos\theta_0 - \frac{m\omega_0^2\sin^2\theta_0}{2}r^2.$

d) Eq $\Rightarrow \left(\frac{d\mathcal{E}_p}{dr}\right)_{r_e} = 0 = mg\cos\theta_0 - m\omega_0^2\sin^2\theta_0\,r_e$

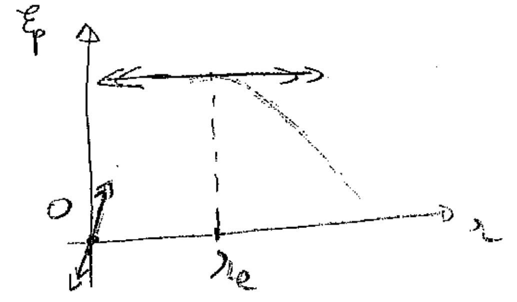
$$\Rightarrow r_e = \frac{g\cos\theta_0}{\omega_0^2\sin^2\theta_0}$$

Eq stable $\Leftrightarrow \left(\frac{d^2\mathcal{E}_p}{dr^2}\right)_{r_e} > 0.$

Or, $\left(\frac{d^2\mathcal{E}_p}{dr^2}\right)_{r_e} = -(m\underbrace{\omega_0^2\sin^2\theta_0}_{cte}) < 0$

$\Rightarrow$ re est une position d'éq instable.

e)



$\left(\frac{d\mathcal{E}_p}{dr}\right)_{r=0} = mg\cos\theta_0 > 0$

Donc, si $r_0 < r_e$, la masse m retombe en O.
si $r_0 > r_e$, la masse m remonte la tige, jusqu'à sortir.

(de manière à tendre vers 1 énergie minimale).

Exercice II

1- $\mathcal{R}_0 = (T, \vec{e}_{x_0}, \vec{e}_{y_0}, \vec{e}_{z_0})$ axes fixes (3 étoiles) / Képler ou Copernic

2- Th de Gauss : $\oiint_{(\Sigma)} \vec{g} \cdot \vec{n}\, dS = -4\pi G M_{int\,à\,(\Sigma)}$ et $\vec{g}(r) = -g(r)\,\vec{e}_r$

(Σ) → sphère de centre T et rayon r

$\Rightarrow -g(r) \cdot 4\pi r^2 = -4\pi G M \Leftrightarrow g(r) = +\frac{GM}{r^2} = +\frac{GM}{(R+z)^2}$

$g_0 = g(R) = +\frac{GM}{R^2} \Rightarrow GM = g_0 R^2$

$\Rightarrow g(r) = g_0 \frac{R^2}{(R+z)^2}$

2- $\Sigma \vec{F} = m\vec{a} = -m\frac{V^2}{r}\vec{e}_r + m\frac{dV}{dt}\vec{e}_\theta$

$\Leftrightarrow -\frac{GmM}{r^2}\vec{e}_r = -\frac{mV^2}{r}\vec{e}_r$ et $\frac{dV}{dt} = 0 \Rightarrow V = \sqrt{\frac{GM}{r}} = \boxed{\sqrt{\frac{g_0 R^2}{r}}}$

3- $T = \frac{2\pi r}{V} = \frac{2\pi}{R}\sqrt{\frac{r^3}{g_0}}$

4- $\mathcal{E}_k = \frac{1}{2}mV^2 = \frac{1}{2}\frac{GmM}{r} = \frac{1}{2}m g_0 \frac{R^2}{r}$, $\mathcal{E}_p = -\frac{GmM}{r} = -m g_0 \frac{R^2}{r} = -2\mathcal{E}_k$

5- $\mathcal{E}_m = \mathcal{E}_k + \mathcal{E}_p = -\frac{1}{2} m g_0 \frac{R^2}{r} < 0 \Rightarrow$ état lié.

6- Géostationnaire s'il est tjs à la verticale du m pt de la Terre.